Engineering Notes

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A Subsonic Nonplanar Kernel Function for Surfaces Inclined to the Freestream

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Introduction

As was pointed out by Ashley et al.,[1,2] the subsonic kernel function, originally, derived by Küssner[3] and further developed by Watkins et al.,[4,5] is not inherently restricted to planar wings, but can be extended to more general multiple-surface configurations. Further, Ashley et al.,[1,2] presented a partial extension of this function for a surface with spanwise curvature whose chord is everywhere parallel to the freestream direction. Subsequently, similar kernel functions for surfaces with spanwise curvature were presented in a form more amenable to machine computation.[6-8]

In this Note, the subsonic kernel function is generalized even further to admit surfaces whose chords are inclined to the freestream. Although the nonplanar nature of the kernel function does not extend to chordwise curvature, it does include surfaces whose chords can be represented by straight-line segments, since each segment may be considered analytically as a separate surface. Thus, one possible application of this extended kernel function is a wing with a deflected control surface. It is noted that this extension is a linear approximation to an essentially nonlinear condition, so its applicability is limited to surfaces with small inclinations to the freestream.

General Nonplanar Kernel Function

Consider a surface of zero thickness $S(x,y,z)$ in a freestream of velocity U and Mach number M undergoing harmonic oscillations with frequency ω. The kernel function defining the downwash at a point (x,y,z) due to a harmonically oscillating pressure doublet of unit strength at a point (ξ,η,ζ) may be written in integral form as

$$K(x_0,y_0,z_0) = \lim_{\substack{x,y,z\to S\\ \xi,\eta,\zeta\to S}} \left[\frac{\partial}{\partial n}\left\{\exp\left[-i\omega\frac{x_0}{U}\right]\int_{-\infty}^{x_0}\frac{\partial}{\partial n_1}\times \left\{\frac{1}{R}\exp\left[i\frac{\omega}{U\beta^2}(\lambda - MR)\right]\right\}d\lambda\right\}\right] \quad (1)$$

where $x_0 = x - \xi$, $y_0 = y - \eta$, $z_0 = z - \zeta$, n denotes the normal to the surface at (x,y,z), n_1 denotes the normal to the surface at (ξ,η,ζ), $\beta = (1 - M^2)^{1/2}$, and $R = [\lambda^2 + \beta^2(y-\eta)^2 + \beta^2(z-\zeta)^2]^{1/2}$.

The normal derivative $\partial/\partial n$ may be written as

$$\partial/\partial n = f_1(x,y,z)(\partial/\partial z) + f_2(x,y,z)(\partial/\partial y) + f_3(x,y,z)(\partial/\partial x) \quad (2)$$

Similarly,

$$\partial/\partial n_1 = g_1(\xi,\eta,\zeta)(\partial/\partial\zeta) + g_2(\xi,\eta,\zeta)(\partial/\partial\eta) + g_3(\xi,\eta,\zeta)(\partial/\partial\xi) \quad (3)$$

After substituting Eq. (3) into Eq. (1) bearing in mind that $\partial/\partial x_0 = -\partial/\partial\xi$, the following result is obtained:

$$K(x_0,y_0,z_0) = \lim_{\substack{x,y,z\to S\\ \xi,\eta,\zeta\to S}} \left[\frac{\partial}{\partial n}\left\{\exp\left[-i\omega\frac{x_0}{U}\right]\int_{-\infty}^{x_0}\times \left\{-g_3(\xi,\eta,\zeta)\frac{\partial}{\partial\lambda} + g_2(\xi,\eta,\zeta)\frac{\partial}{\partial n} + g_1(\xi,\eta,\zeta)\frac{\partial}{\partial\zeta}\right\}\times \left\{\frac{1}{R}\exp\left[i\bar{\omega}(\lambda - MR)\right]\right\}d\lambda\right\}\right] \quad (4)$$

where

$$\bar{\omega} = \omega/U\beta^2 \quad (5)$$

Examination of Eq. (4) indicates that considerable simplification may be introduced, with only a small sacrifice in potential usefulness, by restricting consideration to surfaces for which the $g_i(\xi,\eta,\zeta)$ are not functions of ξ. With this restriction, the kernel function is now written as

$$K(x_0,y_0,z_0) = \lim_{\substack{x,y,z\to S\\ \xi,\eta,\zeta\to S}} \left[\frac{\partial}{\partial n}\left\{\exp\left[-i\omega\frac{x_0}{U}\right]\times\left[-g_3(\eta,\zeta)\times \frac{1}{R'}\exp\left[i\bar{\omega}(x_0 - MR')\right]\right] + \left[g_2(\eta,\zeta)\frac{\partial}{\partial\eta} + g_1(\eta,\zeta)\frac{\partial}{\partial\zeta}\right]\times \int_{-\infty}^{x_0}\frac{1}{R}\exp\left[i\bar{\omega}(\lambda - MR)\right]d\lambda\right\}\right] \quad (6)$$

Further manipulation in the manner of Ref. 4 yields the following final expression for the kernel function:

$$K(x_0,y_0,z_0) = \exp\left[-i\omega\frac{x_0}{U}\right]\cdot\sum_{i=1}^{3}\sum_{j=1}^{3}A_{ij}f_i g_j \quad (7)$$

where the f_i and g_j are defined by Eqs. (2) and (3), $r = \beta(y_0^2 + z_0^2)^{1/2}$, and, employing $(\bar{\omega}\beta r)$ as the argument of the modified Bessel and Struve functions, K_i, I_i, L_i,

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$$
\begin{aligned}
A_{11} = {} & \frac{\beta\omega}{Ur}\left[K_1 + \frac{i\pi}{2}(I_1 - L_1)\right] + \left[\frac{\beta\omega}{Ur}\right]^2 z_0^2\left[\frac{i\pi}{2}(I_2 - L_2) - K_2\right] + \\
& \left[\frac{\beta\omega}{Ur}\right]^2 z_0^2 \int_0^{[x_0 - M(x_0^2+r^2)^{1/2}]/\beta r} \tau^2(1+\tau^2)^{-1/2}\exp[i\bar{\omega}r\beta\tau]d\tau - \frac{i}{3}\left[\frac{\omega}{U}\right]^3 \frac{\beta z_0^2}{r} - \\
& i\,\frac{\beta\omega}{Ur}\left[\frac{\beta y_0}{r}\right]^2 \int_0^{[x_0 - M(x_0^2+r^2)^{1/2}]/\beta r} \tau(1+\tau^2)^{-1/2}\exp[i\bar{\omega}r\beta\tau]d\tau - i\,\frac{\omega}{U}\,\frac{\beta^3 y_0^2}{r_3} + \\
& \exp\left\{i\bar{\omega}[x_0 - M(x_0^2+r^2)^{1/2}]\right\}\left\{\frac{x_0^3(y_0^2 - z_0^2) + x_0r^2(y_0^2 - 2z_0^2)}{(y_0^2+z_0^2)^2(r^2+x_0^2)^{3/2}} + \right. \\
& \left. \frac{i\bar{\omega}x_0z_0^2}{(r^2+x_0^2)(y_0^2+z_0^2)^2}\,[x_0(x_0^2+r^2)^{1/2} - M(x_0^2+2r^2)]\right\} \qquad \text{(8a)}
\end{aligned}
$$

$$
\begin{aligned}
A_{12} = {} & \left[\frac{\beta\omega}{Ur}\right]^2 y_0z_0\left[K_2 - \frac{i\pi}{2}(I_2 - L_2)\right] + \frac{i}{3}\left[\frac{\omega}{U}\right]^3 \frac{\beta y_0z_0}{r} - i\,\frac{\omega}{U}\,\frac{\beta^3 y_0z_0}{r^3} - \\
& i\,\frac{\omega}{U}\,y_0z_0\,\frac{\beta^3}{r^3}\int_0^{[x_0 - M(x_0^2+r^2)^{1/2}]/\beta r} \tau(1+\tau^2)^{-1/2}\exp[i\bar{\omega}r\beta\tau]d\tau - \\
& \left[\frac{\omega\beta}{Ur}\right]^2 y_0z_0\int_0^{[x_0 - M(x_0^2+r^2)^{1/2}]/\beta r} \tau^2(1+\tau^2)^{-1/2}\exp[i\bar{\omega}r\beta\tau]d\tau + \\
& \exp\left\{i\bar{\omega}[x_0 - M(r^2+x_0^2)^{1/2}]\right\}\left\{\frac{x_0y_0z_0(3r^2+2x_0^2)}{(y_0^2+z_0^2)^2(r^2+x_0^2)^{3/2}} + \right. \\
& \left. \frac{i\bar{\omega}x_0y_0z_0}{(r^2+x_0^2)(y_0^2+z_0^2)^2}\,[M(x_0^2+2r^2) - x_0(x_0^2+r^2)^{1/2}]\right\} \qquad \text{(8b)}
\end{aligned}
$$

$$
\begin{aligned}
A_{13} = {} & \left[\frac{\omega}{U}\right]^2 \frac{Bz_0}{r}\int_0^{[x_0 - M(x_0^2+r^2)^{1/2}]/\beta r} \tau(1+\tau^2)^{-1/2}\exp[i\bar{\omega}r\beta\tau]d\tau + \exp\{i\bar{\omega}[x_0 - M(r^2+x_0^2)^{1/2}]\}\times \\
& \left\{-\frac{\beta^2 z_0}{(r^2+x_0^2)^{3/2}} + \frac{i\bar{\omega}\beta^2 z_0[x_0(x_0^2+r^2)^{1/2} - M(y_0^2+z_0^2)]}{(r^2+x_0^2)(y_0^2+z_0^2)}\right\} + \\
& \left[\frac{\omega}{U}\right]^2 \frac{\beta z_0}{r} + i\left[\frac{\omega}{U}\right]^2 \frac{\beta z_0}{r}\left[K_1 + \frac{i\pi}{2}(I_1 - L_1)\right] \qquad \text{(8c)}
\end{aligned}
$$

$$
A_{21} = A_{12} \qquad \text{(8d)}
$$

$$
\begin{aligned}
A_{22} = {} & \frac{\beta\omega}{Ur}\left[K_1 + \frac{i\pi}{2}(I_1 - L_1)\right] + \left[\frac{\beta\omega}{Ur}\right]^2 y_0^2\left[\frac{i\pi}{2}(I_2 - L_2) - K_2\right] + \\
& \left[\frac{\beta\omega}{Ur}\right]^2 y_0^2 \int_0^{[x_0 - M(x_0^2+r^2)^{1/2}]/\beta r} \tau^2(1+\tau^2)^{-1/2}\exp[i\bar{\omega}r\beta\tau]d\tau - \\
& \frac{i}{3}\left[\frac{\omega}{U}\right]^3 \frac{\beta y_0^2}{r} - \frac{i\beta\omega}{Ur}\left[\frac{\beta z_0}{r}\right]^2 \int_0^{[x_0 - M(x_0^2+r^2)^{1/2}]/\beta r} \tau(1+\tau^2)^{-1/2}\exp[i\bar{\omega}r\beta\tau]d\tau - \\
& \frac{i\omega}{U}\,\frac{\beta^3 z_0^2}{r^3} + \exp\{i\bar{\omega}[x_0 - M(x_0^2+r^2)^{1/2}]\}\times\left\{\frac{x_0^3(z_0^2 - y_0^2) + x_0r^2(z_0^2 - 2y_0^2)}{(y_0^2+z_0^2)^2(r^2+x_0^2)^{3/2}} + \right. \\
& \left. \frac{i\bar{\omega}x_0y_0^2}{(r^2+x_0^2)(y_0^2+z_0^2)^2}\,[x_0(x_0^2+r^2)^{1/2} - M(x_0^2+2r^2)]\right\} \qquad \text{(8e)}
\end{aligned}
$$

$$
\begin{aligned}
A_{23} = {} & -\beta\left[\frac{\omega}{U}\right]^2 \frac{y_0}{r}\int_0^{[x_0 - M(x_0^2+r^2)^{1/2}]/\beta r} \tau(1+\tau^2)^{-1/2}\exp[i\bar{\omega}r\beta\tau]d\tau + \\
& \exp\{i\bar{\omega}[x_0 - M(r^2+x_0^2)^{1/2}]\}\left\{\beta^2 y_0(r^2+x_0^2)^{-3/2} - \frac{i\bar{\omega}\beta^2 y_0[x_0(r^2+x_0^2)^{1/2} - M(y_0^2+z_0^2)]}{(y_0^2+z_0^2)(r^2+x_0^2)}\right\} - \\
& \left[\frac{\omega}{U}\right]^2 \frac{\beta y_0}{r} - i\left[\frac{\omega}{U}\right]^2 \frac{\beta y_0}{r}\left[K_1 + \frac{i\pi}{2}(I_1 - L_1)\right] \qquad \text{(8f)}
\end{aligned}
$$

$$
A_{31} = \exp\{i\bar{\omega}[x_0 - M(r^2+x_0^2)^{1/2}]\}\times\left\{-iM\,\frac{\omega}{U}\,z_0(r^2+x_0^2)^{-1} - \beta^2 z_0(r^2+x_0^2)^{-3/2}\right\} \qquad \text{(8g)}
$$

$$
A_{32} = \exp\{i\bar{\omega}[x_0 - M(r^2+x_0^2)^{1/2}]\}\times\left\{iM\,\frac{\omega}{U}\,y_0(r^2+x_0^2)^{-1} + \beta^2 y_0(r^2+x_0^2)^{-3/2}\right\} \qquad \text{(8h)}
$$

$$
A_{33} = \exp\{i\bar{\omega}[x_0 - M(r^2+x_0^2)^{1/2}]\}\times\left\{iM\,\frac{\omega}{U\beta^2}\,\frac{x_0 - M(r^2+x_0^2)^{1/2}}{r^2+x_0^2} + x_0(r^2+x_0^2)^{3/2}\right\} \qquad \text{(8i)}
$$

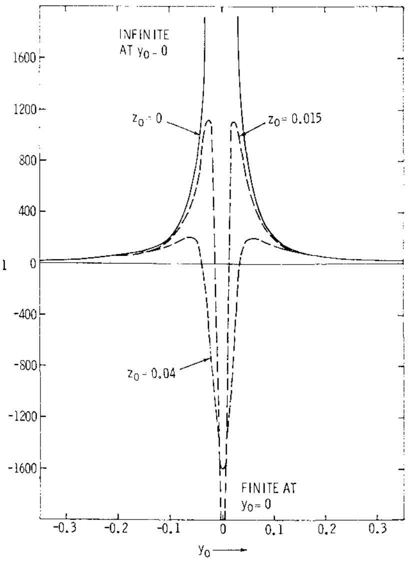

Fig. 1 Comparison of nonplanar and planar kernel function term A_{11} for $M = 0.6$, $\omega = 0$.

For the steady case ($\omega = 0$), the kernel function expressions reduce to

$$A_{11} = \frac{1}{y_0^2 + z_0^2} - \frac{2z_0^2}{(y_0^2 + z_0^2)^2} + \frac{x_0^3(y_0^2 - z_0^2) + x_0 r^2(y_0^2 - 2z_0^2)}{(y_0^2 + z_0^2)^2(r^2 + x_0^2)^{3/2}} \tag{9a}$$

$$A_{12} = \frac{2y_0 z_0}{(y_0^2 + z_0^2)^2} + \frac{x_0 y_0 z_0(2x_0^2 + 3r^2)}{(y_0^2 + z_0^2)^2\,(r^2 + x_0^2)^{3/2}} \tag{9b}$$

$$A_{13} = -\beta^2 z_0/(r^2 + x_0^2)^{3/2} \tag{9c}$$

$$A_{21} = A_{12} \tag{9d}$$

$$A_{22} = \frac{1}{y_0^2 + z_0^2} - \frac{2y_0^2}{(y_0^2 + z_0^2)^2} + \frac{x_0^3(z_0^2 - y_0^2) + x_0 r^2(z_0^2 - 2y_0^2)}{(y_0^2 + z_0^2)^2(r^2 + x_0^2)^{3/2}} \tag{9e}$$

$$A_{23} = \beta^2 y_0/(r^2 + x_0^2)^{3/2} \tag{9f}$$

$$A_{31} = A_{13} \tag{9g}$$

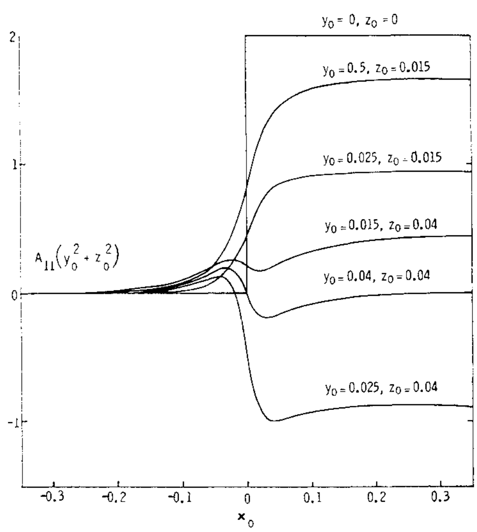

Fig. 2 A_{11} $(y_0^2 + z_0^2)$ vs x_0 for selected values of y_0 and z_0, $\omega = 0$.

$$A_{32} = A_{23} \tag{9h}$$

$$A_{33} = x_0/(r^2 + x_0^2)^{3/2} \tag{9i}$$

Concluding Remarks

It can be seen that the nonplanar kernel function $K(x_0,y_0,z_0)$ possesses no terms that are not numerically similar to those of the original planar kernel function.[4] Consequently, it will be equally tractable to numerical evaluation.

Figure 1 shows a spanwise plot of the A_{11} term (Eq. 9a) of the nonplanar kernel function for large positive x_0 contrasted with the planar kernel function. To better describe the nature of the nonplanar kernel function, it is again plotted in Fig. 2, now against x_0, thereby revealing the chordwise variation. Here, however, A_{11} has been multiplied by the factor $(y_0^2 + z_0^2)$ to remove the singularity at $y_0 = z_0 = 0$. Taking this factor into account, the curves of Fig. 1 represent cross-sectional cuts passing through the far right-hand portion of the plots of Fig. 2. It is seen that the variation of the nonplanar kernel function in the neighborhood of the origin is considerably more complex than that of the planar kernel. It is noted also that, whereas Watkins et al.[5] utilized the Mangler "finite part of the integral" concept in the spanwise integration to obtain pressure distributions, this concept is not applicable to the integration of the nonplanar kernel function ($z_0 \neq 0$),§ since it is nonsingular.

The expressions of Eqs. (7) and (8) readily reduce to Ashley's[2] expression for the nonplanar kernel function in compressible flow if the chord is set parallel to the freestream. For this case

$$f_1(x,y) = \cos\theta(x,y);\ f_2(x,y) = -\sin\theta(x,y)$$
$$g_1(\xi,\eta) = \cos\theta(\xi,\eta);\ g_2(\xi,\eta) = -\sin\theta(\xi,\eta)$$

It can also be demonstrated that the expressions herein reduce to those of Ref. 7 when appropriate coordinate system transformations are made.

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§ For $z_0 = 0$, the nonplanar kernel function is actually a planar kernel function, so of course for integrations through a point $z_0 = 0$ the Mangler approach must be used.